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CONTENTS

MUSICAL MODULATION BY SYMMETRIES

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The symmetry-oriented model of musical modulation developed by G. Mazzola (1985, 1990) is applied to arbitrary scales of seven different tones in twelve-part equal-tempered tuning. For each of the sixty-six different translation-classes of such scales, all possible transitions have been computer generated according to the model. A comparative evaluation of the data exhibits special characteristics for the most common musical scales: the diatonic major scale, the melodic minor scale and the harmonic minor scale.

I. Introduction

Recently the concept of symmetry has been applied to music theoretical problems with success (Mazzola 1990). For example, counterpoint and the theory of modulation pose challenges for mathematical music theory. In both cases, modeling analogous to that of modern physics, using symmetries to explain transitional forces, leads to questions about the existence of local symmetries in counterpoint (Mazzola 1989) and of quanta in modulation theory (see "Modulationsquant" [Mazzola, 1990, 200–01]). And in both cases one can interpret results as generalizations of classical music theory.

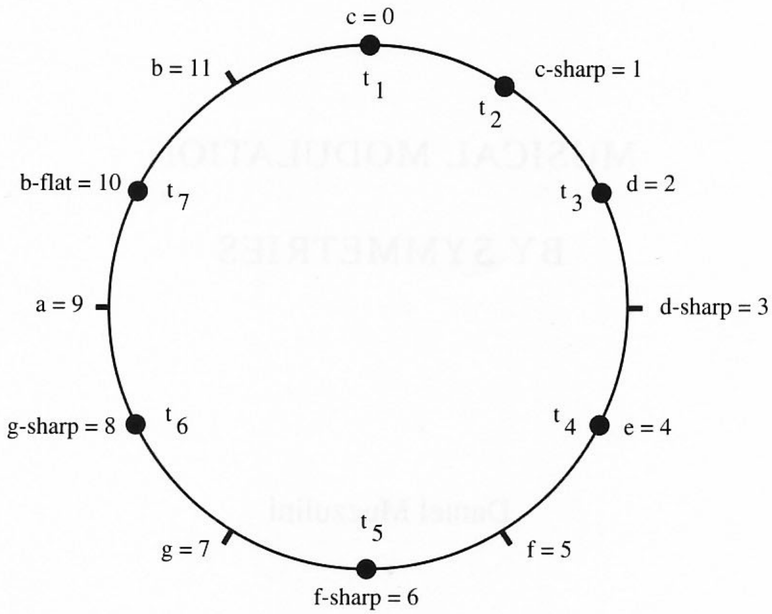

Figure 1

The modulation model developed in Mazzola 1985 provides direct modulations (for all degrees of relationship in the cycle of fifths) between the most common musical scales in 12-part equal-tempered tuning, and the obtained modulations for the classical degrees of relationship agree with classical theory (Schoenberg 1911). A similar model has been developed for just tuning (Mazzola 1990), but we adhere to the equal-tempered tuning in this article. Here, we extend the model to include arbitrary scales consisting of seven tones, thus providing a harmonic basis for the use of uncommon musical scales in composition and a point of departure for further investigation.

II. Triadic Interpretation of Scales

We interpret a *scale s* of seven different pitch-classes as a (seven-element) subset of the cyclic group $\mathbf{Z}_{12}$. We view $\mathbf{Z}_{12}$ as being the set of pitch-classes defining the 12-tempered chromatic scale. To simplify, we shall speak of "tones" instead of "pitch-classes." Tones will be denoted

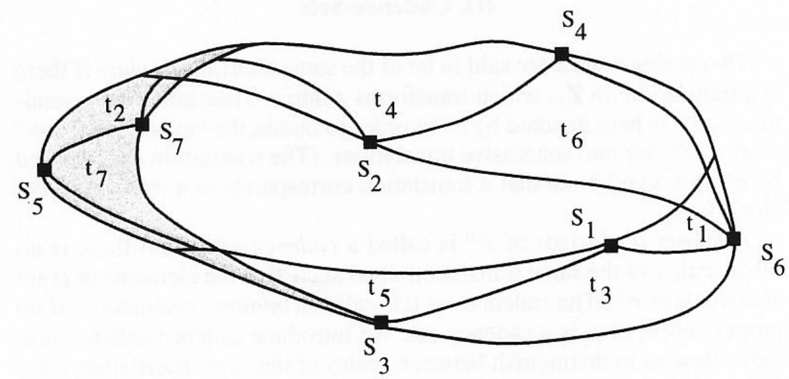

Figure 2

as c = 0, c♯ = 1, d = 2, and so forth. We can represent $\mathbf{Z}_{12}$ as the circumference of a clock-face. Starting with an arbitrary tone t_1, the "tonic" of s, we number the tones of s in clockwise direction: $t_1, t_2, \ldots, t_7$. Figure 1 shows a scale (No. 62 from the Appendix in Mazzola 1990) consisting of seven tones marked by dots and numbered in clockwise direction.

For $n = 1$ to 7, we define the *triad on the n-th degree* of s as being the set s_n containing the three tones t_n, $t_{(n+2)\ mod\ 7}$, and $t_{(n+4)\ mod\ 7}$. Thus $s_1 = \{t_1, t_3, t_5\}$, $s_2 = \{t_2, t_4, t_6\}$, ... , $s_7 = \{t_7, t_2, t_4\}$. The covering of s by the seven triads $s_1, s_2, \ldots, s_7$ is called the *triadic interpretation* of s and will be denoted $s^{(3)}$ according to Mazzola 1990. Clearly, $s^{(3)}$ is independent of the choice of the tonic. The "nerve" (Mazzola 1990) of a triadic interpretation $s^{(3)}$ of a scale s is a Möbius strip (Figure 2). The nerve is constructed as follows: The triads $s_1, s_2, \ldots, s_7$ of $s^{(3)}$ are represented as dots. The dots of two triads are connected by a line if their triads have a non-empty intersection. Three triads with a non-empty intersection are connected by a triangle. Four triads with non-empty intersection would build up a tetrahedron and so forth. The construction may be applied to arbitrary coverings of sets (Mazzola 1990).

The triadic interpretation of "exotic" scales can also be deduced by altering the canonical triadic interpretation of diatonic scales; the alterations must be made in such a way that no tone crossings occur (e.g., not altering e to e♯ and f to f♭ at the same time) so as to preserve the ordered numbering of the tones.

III. Cadence-Sets

Two scales r and s are said to be of the same *translation-class* if there is a translation on $\mathbf{Z}_{12}$ which transforms r into s. Translation by p semitone-steps is here denoted by e^p in order to obtain the "exponential law" $e^p \cdot e^q = e^{p+q}$ for two successive translations. (The translation e^p is defined by $e^p(x) = x+p$.) Note that a translation corresponds to a rotation of the clock-face.

A subset μ of triads of $s^{(3)}$ is called a *cadence-set* of s if there is no other scale r of the same translation-class such that the elements of μ are also triads of $r^{(3)}$. The cadence-set μ is called a *minimal cadence-set* if no proper subset of μ is a cadence-set. We introduce cadence-sets because they allow us to distinguish between scales of the same translation-class without having to enumerate all of their tones or triads (Mazzola, 1990).

Examples

(As usual we denote the triads of the diatonic scales with roman numerals, and we index the key if necessary. For instance, IV_C denotes the triad of the fourth degree of C-major.)

1. A diatonic major scale has the following minimal cadence-sets: {II, III}, {III, IV}, {IV, V}, {II, V}, {VII}. The set {I, V} is not a cadence-set, for $I_C = IV_G$ and $V_C = I_G$. The set {I, IV, V} is a cadence-set but not a minimal one.
2. For the harmonic minor scale, each pair of triads from its triadic interpretation forms a minimal cadence-set. Therefore, there are 21 different minimal cadence-sets for a harmonic minor scale.

We have generated by computer the cadence-sets of the 66 translation-classes. They will be used to calculate modulations between the scales of a given translation-class. The calculus of cadence-sets may be restricted to representatives of the 38 seven-tone scale-orbits under the action of the group of translations and inversions. Appendix 1 lists these 38 scale-orbits representatives and their respective numbers of minimal cadence-sets. (The scale-orbit numbering follows the more complete listing in Mazzola 1990.) The number of minimal cadence-sets (given in column 2) varies between 5 and 21. Only the harmonic minor scale (No. 54.1) has 21 minimal cadence-sets. Next most plentiful are the cadence-sets of scale-orbit No. 58 with 18 minimal cadence-sets, and scale-orbit No. 47.1 (the melodic minor scale) with 15 minimal cadence-sets. There are three scale-orbits with only 5 minimal cadence-sets: No. 38.1 (the diatonic major scale), No. 52, and No. 62.

IV. Inner Symmetries of Triadic Interpretations

In the present context, invertible affine transformations on $\mathbf{Z}_{12}$ are called *symmetries.* They are written in the form $e^p u$ with p in $\mathbf{Z}_{12}$ and u invertible in $\mathbf{Z}_{12}$ (u=1,5,7, or 11) and defined by $e^p u(x)=p+ux$. The *inner symmetries* of a scale s are the symmetries leaving the set s invariant. They define the *symmetry group* of s. A scale with trivial symmetry group is called *rigid.*

Examples

1. The C-major-scale {0, 2, 4, 5, 7, 9, 11} has a unique non-trivial inner symmetry $e^4 11$, the inversion at d=2.
2. The harmonic minor scales are rigid.
3. The scale {0, 1, 2, 4, 6, 8, 10} has the following three non-trivial inner symmetries: $e^8 5$, $e^6 7$ and $e^2 11$.

A classification of all subsets of $\mathbf{Z}_{12}$ under the action of the full group of symmetries is available in Mazzola (1985). For the scales of seven tones we may refer to Appendix 1.

The concept of inner symmetries may be carried over to triadic interpretations (Mazzola 1990). An inner symmetry f on s is called an *inner symmetry of the triadic interpretation* $s^{(3)}$, if for each s_i in $s^{(3)}$, $f(s_i)$ is also a triad of $s^{(3)}$. A triadic interpretation $s^{(3)}$ is termed *rigid,* if the symmetry group of $s^{(3)}$ is trivial. We have the following results:

Lemma

(1) *Any inversion in the symmetry group of a scale s induces a symmetry of* $s^{(3)}$.

(2) *The only possible non-trivial inner symmetries of triadic interpretations are inversions.*

In particular, a triadic interpretation $s^{(3)}$ is rigid, if and only if the symmetry group of s contains no inversion. (The inversions take the form $e^p u$, where u=11.)

Examples

1. The inner symmetry $e^4 11$ of the C-major-scale C operates on the triads of $C^{(3)}$ as follows:

I	→	VI	VI	→	I
III	→	IV	IV	→	III
V	→	II	II	→	V
VII	→	VII			

2. The inner symmetry $e^8 5$ of the scale $s = \{0, 1, 2, 4, 6, 8, 10\}$ is not an inner symmetry of $s^{(3)}$, since the image of $s_1 = \{0, 2, 6\}$ under $e^8 5$ is $\{8, 6, 2\}$, which does not belong to $s^{(3)}$.
3. Property (1) would no longer be valid if arbitrary numberings of the tones of scales were used to construct more general "triadic interpretations" of scales. For instance, number the tones of C-major as follows: t_1=0, t_2=2, t_3=5, t_4=4, t_5=7, t_6=9, t_7=11. Then the inner symmetry of $e^4 11$ maps $s_1=\{0, 5, 7\}$ to $\{4, 11, 9\}=\{t_4, t_6, t_7\}$, which is not a triad in this "interpretation" of C-major.

V. The Concept of Modulation in the Light of Symmetries

Arnold Schoenberg (1911) defines modulation as a tripartite process:

A	B	C
old key	new key	new key
neutral triads	**pivot root-progressions** (in	**cadence**
to weaken the	German *Fundamentschritte*)	to establish
old key	to mark the turning point	the new key

It is of special interest to determine suitable pivot-progressions for a given pair of scales, s and $r=e^p(s)$. By analogy with particle-physics, we will interpret modulations by hidden symmetries, which are supported by a "quantum" (see "Modulationsquant" [Mazzola 1990, 200–01]). Following Mazzola 1990, the explicit construction of the quantum will permit the calculation of the pivot-progressions.

We define a *modulator* for the pair $(s^{(3)},r^{(3)})$, where $r=e^p(s)$, to be a symmetry g which transforms the triadic interpretation $s^{(3)}$ into the triadic interpretation $r^{(3)}$. Modulators can be written in the form $g=e^p f$, where f is an inner symmetry of $s^{(3)}$. Hence, by statement (2) of the Lemma the only candidates for modulators are translations and inversions.

Fix a minimal cadence-set μ of the target scale r. This being done, we propose a system (£$_\mu$) of properties that define particular subsets Q of $\mathbf{Z}_{12}$, which we call the *modulation-quanta:*

(£$_\mu$)

- (1) There is a modulator g for $s^{(3)}$ and $r^{(3)}$ which is an inner symmetry of Q.
- (2)$_\mu$ All triads of μ are subsets of Q.
- (3) The only inner symmetry of $r \cap Q$ of the form $e^p u$, for u = 1 or 11 (translations or inversions), is the identity, and $r \cap Q$ is covered by triads of $r^{(3)}$.
- (4) Q is a minimal set with properties (1) and (2)$_\mu$.

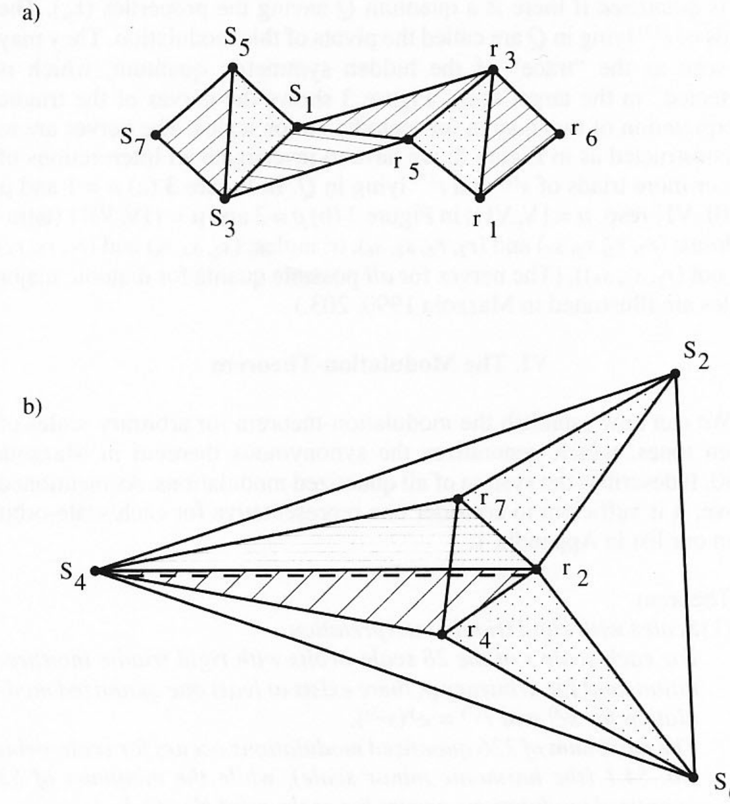

Figure 3

The main problem with this modulation model is the question of the existence of quanta. But let us first discuss their properties in detail, in order to understand this conceptual construction.

Point (1) requires that the modulator is materialized in Q. Point (2)$_\mu$ guarantees that Q has enough tones to capture $r^{(3)}$ uniquely by the cadence μ. Point (4) expresses our interest in finding the most economical modulations. Because of (3), the modulator from (1) is uniquely determined by Q and the pair $(s^{(3)}, r^{(3)})$. On the other hand Q is reconstructible from the modulator and the triads of $r^{(3)}$ lying in Q, since $r \cap Q$ is covered by triads of $r^{(3)}$ and because of the minimality of Q.

The pair (μ,g) of minimal cadence-set μ and modulator g is called a *modulation* from $s^{(3)}$ to $r^{(3)}$. We say that a modulation (μ,g) from $s^{(3)}$ to

$r^{(3)}$ is *quantized* if there is a quantum Q having the properties ($£_\mu$). The triads of $r^{(3)}$ lying in Q are called the pivots of this modulation. They may be seen as the "trace" of the hidden symmetric quantum, which is "detected" in the target scale. Figure 3 shows the nerves of the triadic interpretation of two quanta for melodic minor scales. The nerves are to be constructed as in Figure 2; we have to investigate all intersections of two or more triads of $s^{(3)}$ and $r^{(3)}$ lying in Q. In Figure 3 (a) $p = 1$ and $\mu = \{\mathrm{III}, \mathrm{VI}\}$ resp. $\mu = \{\mathrm{V}, \mathrm{VI}\}$; in Figure 3 (b) $p = 3$ and $\mu = \{\mathrm{IV}, \mathrm{VII}\}$ (tetrahedrons: (r_7, r_2, s_2, s_4) and (r_2, r_4, s_4, s_6); triangles: (s_2, s_4, s_6) and (r_7, r_2, r_4) but not (r_2, s_2, s_6)). (The nerves for *all* possible quanta for diatonic major scales are illustrated in Mazzola 1990, 203.)

VI. The Modulation-Theorem

We can now establish the modulation-theorem for arbitrary scales of seven tones, which generalizes the synonymous theorem in Mazzola 1990. It describes the system of all quantized modulations. As mentioned above, it is sufficient to consider one representative for each scale-orbit from our list in Appendix 1.

Theorem

(1) *Scales with rigid triadic interpretation:*

For each scale s of the 28 scale-orbits with rigid triadic interpretation, and for arbitrary p, there exists at least one quantized modulation for $s^{(3)}$ and $r^{(3)} = e^p(s^{(3)})$.

The maximum of 226 quantized modulations occurs for scale-orbit No. 54.1 (the harmonic minor scale), while the minimum of 53 quantized modulations occurs for scale-orbit No. 41.1

(2) *Scales with non-rigid triadic interpretation:*

For scale-orbits No. 52 and No. 55, there exist quantized modulations except for p=1 and p=11. And for scale-orbits No. 38 and No. 62 there exist quantized modulations except for p=5 and p=7. The 6 remaining orbits have at least one quantized modulation for each p.

The maximum of 114 quantized modulations occurs at scale-orbit No. 47.1 (melodic minor scale). Among the scales with quantized modulations for each p the minimum of 26 occurs at scale-orbit No. 38.1 (diatonic major scale).

We have used a computer program to calculate the quanta and pivots as follows: For a given pair ($s^{(3)}$, $r^{(3)} = e^p(s^{(3)})$), choose a corresponding modulation (μ,g). Then there is exactly one subset Q of $\mathbf{Z}_{12}$ which fulfills (1), $(2)_\mu$, and (4) from ($£_\mu$). (Q is the orbit of tones of μ under the group of symmetries generated by g). This candidate is rejected if (3) does not hold, and we may choose a new modulation.

Lists of all quantized modulations for diatonic major scales, melodic minor scales, and harmonic minor scales are given in Appendices 2, 3, and 4 respectively.

VII. Discussion

Different minimal cadence-sets may produce the same quanta for a given p. This is the case for the minimal cadence-sets $\{\mathrm{II}, \mathrm{V}\}$ and $\{\mathrm{IV}, \mathrm{V}\}$ of the diatonic major scale and for $p = 2$. If $s^{(3)}$ is not rigid, the two possible modulators can produce the same pivots with different quanta. In this case the presentation of the pivots is not sufficient to recover the modulator. For the diatonic major scale, this happens for $p = 6$, those modulations which have to surmount the greatest "tonal distance."

Within the groups of rigid (and respectively, non-rigid) triadic interpretations, the number of quantized modulations essentially increases with the number of minimal cadence-sets. Given the number of minimal cadence-sets, the scales with rigid triadic interpretation have generally more quantized modulations than those with non-rigid triadic interpretation.

It must be noted that for rigid triadic interpretations, with $p = 1, 5, 7$ or 11, the same quantum always occurs, the whole set $\mathbf{Z}_{12}$; it is the only non-empty subset of $\mathbf{Z}_{12}$ whose symmetry group includes a translation of e^p for $p = 1, 5, 7$, or 11. Therefore, in the rigid case every cadence-set produces all seven triadic degrees as pivots for $p = 1, 5, 7$, or 11.

For each given scale-orbit, we can add the numbers of different quanta occurring in the quantized modulations for variable p (see column 3 in Appendix 1). Then we obtain the following extremal numbers of quanta:

rigid:	minimum	13	No. 51
	maximum	32	harmonic minor scale (No. 54.1) and No. 54
non-rigid:	minimum	20	diatonic major scale (No. 38.1)
	maximum	66	melodic minor scale (No. 47.1)

Discussion of distinguished scales with rigid triadic interpretation

Scale No. 51 (****0**00*0) has 9 minimal cadence-sets. There exists a quantum only for $p = 4$ and $p = 8$, which does not allow all triads as pivots.

The harmonic minor scale, No. 54.1 (*0**0*0**0*) allows quantized modulations for $p = 1, 5, 7$ and 11, and also for $p = 2, 10$. For these values of p, and for any cadence-set, all seven triads are allowed as pivots. For the remaining values of p, there are several quanta:

$p = 3,9$:	3 quanta, each p
$p = 4, 8$:	7 quanta, each p
$p = 6$:	6 quanta.

The scale No. 54 (****0**00*00), with only 7 minimal cadence-sets, is the "richest" scale with rigid triadic interpretation:

$p = 2, 3, 9, 10$:	3 quanta, each p
$p = 4, 8$:	6 quanta, each p
$p = 6$:	4 quanta.

The triadic interpretation of scale No. 58 (**0**0**0*00) is rigid and has 18 different minimal cadence sets. It is one of the four scale-orbits with no (two) successive half-tone-steps; the others are the diatonic major scale, the melodic and the harmonic minor scale. With the impressive number of 185 quantized modulations, this remarkable scale is second only to the harmonic minor scale. However, it has only 17 quanta, namely 4 different quanta for $p = 4$, and for $p = 8$, and the quantum is $\mathbf{Z}_{12}$ in all other cases.

Discussion of distinguished scales with non-rigid triadic interpretation

The melodic minor scale, No. 47.1 (*0**0*0*0*0*) has 15 minimal cadence-sets and the following numbers of quanta:

$p = 1, 11$:	6, each p
$p = 2, 10$:	3, each p
$p = 3, 5, 6, 7, 9$:	4, each p
$p = 4, 8$:	14, each p

The diatonic major scale optimizes uniqueness of quanta:

$p = 1, 3, 5, 7, 9, 11$:	1 quantum, each p
$p = 2, 10$:	2 quanta, one a subset of the other, each p
$p = 4, 8$:	3 quanta, one in the intersection of the others, each p
$p = 6$:	4 quanta, but only two different sets of fundamentals.

The pivots for the diatonic major scale agree with those proposed by Schoenberg (1911) in all cases in which he indicates direct modulations

(see Mazzola 1985). The model has been applied with success in the analysis of difficult modulations of Mozart, Beethoven and Debussy (Mazzola 1990) and to the composition of a sonata (Mazzola 1985).

VIII. Musicological Context

It must be pointed out that this model of musical modulation need not refer to changes of tonal function of triads, nor to alterations, nor to melodic considerations. This approach realizes a theory of harmony completely restricted to vertical structures. In this respect that model is comparable with the system of Arthur von Oettingen (see Oettingen 1913, Rummenhöller 1967, and Vogel 1966), which is based exclusively on triadic structures. Although Oettingen's conceptual system is completely symmetrical, his use of symmetries to describe musical processes is very restricted. In fact, his fundamental chord progressions can be interpreted as symmetry transitions: the "homonomic fifth-step" as a translation and the "antinomic alternation" as a rotation in the two-dimensional lattice of pitch-classes in just tuning. Modulation itself is not conceived as a symmetry transition. Oettingen instead depends upon theories of triadic tonal function. Hugo Riemann (1905) and Sigfrid Karg-Elert (see Schenk 1966), who adapt some of Oettingen's ideas, also do not pursue this symmetrical concept. Riemann is closer to classical theory in his adumbration of alterations and melodic considerations.

To conclude, we should stress that this model can easily be adapted to different, more general situations, including just tuning, more general interpretations of scales, and microtonal tunings. It is an open question whether this theory can be extended to modulations between scales in different orbits.

NOTES

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Appendix 1
Scale orbits and number of quantized modulations

No.	*s*	(1)	(2)	(3)	(4)	
38	*******00000	$e^6 11$	9	42	54	(!)
38.1	*0*0**0*0*0*	$e^4 11$	5	20	26	
47	*0*****0*000	$e^8 11$	6	28	30	
47.1	*0**0*0*0*0*0	$e^2 11$	15	66	114	
50	**0***0**000	$e^8 11$	7	34	42	
50.1	***0***00*00	$e^6 11$	6	36	46	
52	***0*0***000	$e^8 11$	5	24	24	(!)
55	*****00*0*00	$e^4 11$	6	30	32	(!)
61	***00**0**00	$e^2 11$	10	38	62	
62	***0*0*0*0*0	$e^2 11$	5	24	24	(!)
39	******0*0000		9	29	93	
39.1	*0****0*0*00		6	23	55	
40	*****0**0000		10	24	108	
40.1	***0*0**0*00		7	26	72	
41	****0***0000		7	25	75	
41.1	****0*0**000		6	21	53	
42	******00*000		6	22	54	
42.1	***0**0*0*00		7	28	74	
43	*****0*0*000		6	22	57	
43.1	****0*0*0*00		7	26	72	
44	****0**0*000		9	23	89	
45	***0***0*000		7	21	63	
45.1	**0****0*000		10	21	105	
46	**0****0*000		6	26	56	
48	*****00**000		10	23	109	
48.1	***00***0*00		7	28	68	
49	***0**0**000		7	21	71	
49.1	**00****0*00		7	26	74	
51	****00***000		9	13	86	
53	*****0*00*00		7	27	67	
53.1	*0***0**0*00		9	25	91	
54	****0**00*00		7	32	71	
54.1	*0**0*0**00*		21	32	226	
56	**0***0*0*00		7	24	70	
57	****00**0*00		8	21	71	
58	**0**0**0*00		18	17	185	
59	**0*0***0*00		11	22	101	
60	***0**00**00		6	21	60	

(1) symmetry of $s^{(3)}$; (2) number of μ; (3) number of quanta; (4) number of quantized modulations; (!) not for every *p* quantized.

The numbering of the scale-orbits follows the numbering in [Mazzola 1990]. The scales of the orbits x and x.1 belong to the same orbit under the action of the *full* symmetry group of $\mathbf{Z}_{12}$.

Appendix 2

Quanta and pivots for the modulations between diatonic major scales (No. 38.1)

p	μ	Q	g	pivots
1	{II, V}	*0**0*******	$e^{5}11$	II, III, V, VII
	{II, III}	*0**0*******	$e^{5}11$	II, III, V, VII
2	{VII}	0**0**0*000*	$e^{6}11$	II, IV, VII
	{II, V}	0**0**0*0*0*	$e^{6}11$	II, IV, V, VII
	{IV, V}	0**0**0*0*0*	$e^{6}11$	II, IV, V, VII
3	{II, V}	*0*00*0*****	$e^{7}11$	II, III, V, VII
	{II, III}	*0*00*0*****	$e^{7}11$	II, III, V, VII
4	{VII}	00**0**00*0*	$e^{8}11$	V, VII
	{IV, V}	0*******0*0*	$e^{8}11$	II, IV, V, VII
	{II, III}	****0*****0*	$e^{8}11$	II, III, V, VII
5	{VII}	00*0**0*00**	$e^{9}11$	II, IV, VII
6	{II, III}	0*****0*****	e^{6}	II, III, V, VII
	{IV, V}	0*********0*	$e^{10}11$	II, IV, V, VII
	{IV, V}	****0*****0*	e^{6}	II, IV, V, VII
	{II, III}	****0*0*****	$e^{10}11$	II, III, V, VII
7	{VII}	*0*00**00*0*	$e^{11}11$	III, V, VII
8	{VII}	0**00*0*00**	$e^{0}11$	II, VII
	{IV, V}	0*****0*****	$e^{0}11$	II, IV, V, VII
	{II, III}	****0*0*0***	$e^{0}11$	II, III, V, VII
9	{II, V}	00*0******0*	$e^{1}11$	II, IV, V, VII
	{IV, V}	00*0******0*	$e^{1}11$	II, IV, V, VII
10	{VII}	*0**0*000*0*	$e^{2}11$	III, V, VII
	{II, V}	*0**0*0*0*0*	$e^{2}11$	II, III, V, VII
	{II, III}	*0**0*0*0*0*	$e^{2}11$	II, III, V, VII
11	{II, V}	0**0********	$e^{3}11$	II, IV, V, VII
	{IV, V}	0**0********	$e^{3}11$	II, IV, V, VII

Appendix 3

Quanta and pivots for the modulations between melodic minor scales (No. 47.1)

p	μ	Q	g	pivots
1	{II, IV}, {IV, VII}	****0**00**0	$e^{3}11$	II, IV, VII
	{III, VI}, {V, VI}	******0**0**		I, III, V, VI
	{III, VII}	*00**0****0*		III, V, VII
	{IV, V}	****0******0		II, IV, V, VII
	{II, III}	*00*********		II, III, V, VII
	{I, VII}	*****0****0*		I, III, V, VII
2	{III, V}, {III, VII}, {II, III}	**0***0*0*0*	$e^{4}11$	II, III, V, VII
	{II, IV}, {II, VI}, {I, II}	*0*0**0*0*0*		I, II, IV, VI
	{I, III}, {III, VI}, {III, IV}	0***0*0*0*0*		I, III, IV, VI
3	{III, V}, {III, VI}, {V, VI}, {I, V}	*0**0***00**	$e^{5}11$	I, III, V, VI
	{II, IV}, {IV, VII}	*0**0*00**00		II, IV, VII
	{I, III}	00**00**00**		I, III
	{IV, V}	*0**0*0****0		II, IV, V, VII
4	{III, V}	*00*00**000*	$e^{6}11$	III, V
	{I, III}	00***00*000*		I, III
	{II, VI}, {I, II}	***0****0*0*		I, II, IV, VI
	{IV, VII}	*******00*00		II, IV, VII
	{III, VI}	0*****0*000*		I, III, VI
	{III, VII}	*00*00**0*0*		III, V, VII
	{V, VI}	********000*		I, III, V, VI
	{III, IV}	0*****0*0*0*		I, III, IV, VI
	{II, III}	**0*0***0*0*		II, III, V, VII
	{I, VII}	*0***0**0*0*		I, III, V, VII
	{I, III}	*00**00**00*	e^{4}	I, III
	{III, V}	00**00**00**		III, V
	{III, VII}, {II, III}	0***0***0***		II, III V, VII
	{III, VI}, {III, IV}	**0***0***0*		I, III, IV, VI
5	{I, II}, {I, V}, {III, VI}, {V, VI}	*0****0**00*	$e^{7}11$	I, III, V, VI
	{II, VI}, {I, II}	*0*00*0*****		I, II, IV, VI
	{IV, VII}, {IV, V}	*0****0*0**0		II, IV, V, VII
	{III, VII}	*00**00*****		III, V, VII
6	{III, V}, {III, VII}, {II, III}	**0*0*0***0*	$e^{8}11$	II, III, V, VII
	{I, III}, {III, VI}, {III, IV}	0***0***0*0*		I, III, IV, VI
	{I, III}, {III, VI}, {III, IV}	**0*0***0*0*	e^{6}	I, III, IV, VI
	{III, V}, {III, VII}, {II, III}	0***0*0***0*		II, III, V, VII

7	{III, V}, {I, V}, {III, VII}, {I, VII}	*0**00**0***	$e^{9}11$	I, III, V, VII
	{II, VI}, {I, II}	*0*0**0*0***		I, II, IV, VI
	{IV, VII}, {IV, V}	*0******0*00		II, IV, V, VII
	{III, VI}	00******00**		I, III, VI
8	{III, V}	*00*000*00**	$e^{10}11$	III, V
	{I, III}	00**000**00*		I, III
	{II, VI}	***00*00****		II, IV, VI
	{IV, VII}, {IV, V}	****0*0****0		II, IV, V, VII
	{III, VI}	00**0*0**00*		I, III, VI
	{III, VII}	**0*000*0***		III, V, VII
	{V, VI}	*0**0*0**0**		I, III, V, VI
	{III, IV}	0***0*0***0*		I, III, IV, VI
	{II, III}	**0*0*0*0***		II, III, V, VII
	{I, VII}	****000*****		I, III, V, VII
	{I, III}	*00**00**00*	e^{8}	I, III
	{III, V}	00**00**00**		III, V
	{III, VIII}, {II, III}	0***0***0***		II, III, V, VII
	{III, VI}, {II, IV}	**0***0***0*		I, III, IV, VI
9	{III, V}	*00**00**00*	$e^{11}11$	III, V
	{II, IV}, {II, VI}	*0*00**00*0*		II, IV, VI
	{I, III}, {I, V}, {III, VII}, {I, VII}	*0***00***0*		I, III, V, VII
	{I, II}	*0*0****0*0*		I, III, IV, VI
10	{II, V}, {III, VII}, {II, III}	**0*0*0*0*0*	$e^{0}11$	II, III, V, VII
	{II, IV}, {IV, VII}, {IV, V}	*0**0*0*0**0		II, IV, V, VII
	{I, III}, {III, VI}, {III, IV}	0***0*0*0***		I, III, IV, VI
11	{II, IV}, {II, VI}	***0**00**0*	$e^{1}11$	II, IV, VI
	{III, VI}	00**0****0**		I, III, VI
	{III, VII}, {I, VII}	*****0**0***		I, III, V, VII
	{V, VI}	****0****0**		I, III, V, VI
	{III, IV}	00**********		I, III, IV, VI
	{I, II}	***0******0*		I, II, IV, VI

Appendix 4
Quanta and pivots for the modulations between harmonic minor scales (No. 54.1)

p	μ Number	Q	pivots
3/9	1, 3, 6, 8, 10, 11, 15–20	************	I, II, III, IV, V, VI, VII
	2, 4, 7, 9, 14	*0**0**0**0*	II, IV, VI, VII
	5, 12	0**0**0**0**	II, V, VII
4/8	0, 7, 12, 14–17, 19	************	I, II, III, IV, V, VI, VII
	1, 6, 13	*00**00**00*	I, II, VI
	2	***0***0***0	II, IV
	3	00**00**00**	III, V
	4, 11, 18	**0***0***0*	I, III, IV, VI
	5, 10	0***0***0***	III, V, VII
	8, 20	*0***0***0**	I, III, V, VI
6	2, 7	*0*00**0*00*	II, IV, VII
	3, 10, 17	0***0*0***0*	II, III, V, VII
	4, 9, 14	*0**0**0**0*	II, IV, VI, VII
	5, 12	0**00*0**00*	II, V, VII
	8, 11, 13, 15, 16, 18	****0*****0*	I, II, III, IV, V, VI, VII
	19	***00****00*	II, IV, V, VII
p = 1, 2, 5, 7, 10, 11:		************	I, II, III, IV, V, VI, VII

Numbering of the minimal cadence-sets:

0 : {II, VII}	1 : {I, III}	2 : {II, IV}	3 : {III, V}	4 : {IV, VI}
5 : {V, VII}	6 : {I, VI}	7 : {IV, VII}	8 : {I, V}	9 : {II, VI}
10 : {III, VII}	11 : {I, IV}	12 : {II, V}	13 : {III, VI}	14 : {VI, VII}
15 : {I, VII}	16 : {I, II}	17 : {II, III}	18 : {III, IV}	19 : {IV, V}
20 : {V, VI}				